

Solutions Analysis I

1. Solve the following equations.

(a)

$$\begin{aligned}x^2 - 6x + 8 &= 0 \\x_{1|2} &= \frac{6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 8}}{2 \cdot 1} \\x_1 &= 4 \\x_2 &= 2\end{aligned}$$

(b)

$$\begin{aligned}(3x - 1)^2 - (5x - 3)^2 &= -(4x - 2)^2 \\9x^2 - 6x + 1 - 25x^2 + 30x - 9 &= -16x^2 + 16x - 4 \\8x &= 4 \\x &= 0.5\end{aligned}$$

(c)

$$\begin{aligned}\sqrt{x^2 - 9} &= 9 - x \\x^2 - 9 &= (9 - x)^2 \\x^2 - 9 &= x^2 - 18x + 81 \\18x &= 90 \\x &= 5\end{aligned}$$

(d)

$$\begin{aligned}\log_x(2x + 8) &= 2 \\2x + 8 &= x^2 \\x_{1|2} &= 1 \pm 3\end{aligned}$$

(e)

$$\begin{aligned}e^{2x-5} + 1 &= 4 \\e^{2x-5} &= 3 \\2x - 5 &= \ln 3 \\x &= 0.5 \ln 3 + 2.5\end{aligned}$$

(f)

$$\begin{aligned}\log_2 \frac{2}{x} &= 3 + \log_2 x \\ \frac{2}{x} &= 2^{3+\log_2 x} \\ \frac{2}{x} &= 2^3 \cdot 2^{\log_2 x} \\ \frac{2}{x} &= 8 \cdot x \\ x^2 &= 0.25 \\ x &= 0.5\end{aligned}$$

(g)

$$\begin{aligned}(27)^{2x+1} &= \frac{1}{3} \\ 2x+1 &= \log_{27} \frac{1}{3} \\ 2x+1 &= -\frac{1}{3} \\ x &= -\frac{2}{3}\end{aligned}$$

2. Simplify the following expressions.

(a)

$$\begin{aligned}\frac{4^2 \cdot 6^2}{3^3 \cdot 2^3} &= \frac{(4 \cdot 6)^2}{(3 \cdot 2)^3} \\ &= \frac{24^2}{6^3} \\ &= \frac{1}{6} \cdot \frac{24^2}{6^2} \\ &= \frac{1}{6} 4^2\end{aligned}$$

(b)

$$\begin{aligned}\frac{(x+1)^3(x+1)^{-2}}{(x+1)^2(x+1)^{-3}} &= \frac{(x+1)^3 \cdot (x+1)^3}{(x+1)^2 \cdot (x+1)^2} \\ &= \frac{(x+1)^6}{(x+1)^4} \\ &= (x+1)^2\end{aligned}$$

(c)

$$(-3xy^2)^3 = -27x^3y^6$$

(d)

$$\begin{aligned}\frac{\frac{(x^2)^3}{x^4}}{\left(\frac{x^3}{(x^3)^2}\right)^{-2}} &= \frac{\frac{x^{2 \cdot 3}}{x^4}}{\frac{x^{3 \cdot (-2)}}{x^{3 \cdot 2 \cdot (-2)}}} \\ &= \frac{\frac{x^6}{x^4}}{\frac{x^6}{x^{12}}} \\ &= \frac{x^6}{x^4} \cdot \frac{x^{12}}{x^6} \\ &= \frac{1}{x^4}\end{aligned}$$

(e)

$$\begin{aligned}((2x+1)(2x-1))(4x^2+1) &= (4x^2-2x+2x-1)(4x^2+1) \\ &= (4x^2-1)(4x^2+1) \\ &= 16x^4-1\end{aligned}$$

(f)

$$\begin{aligned}\frac{6x^5+4x^3-1}{2x^2} &= \frac{6x^5+4x^3}{2x^2} - \frac{1}{2x^2} \\ &= 3x^3+2x - \frac{1}{2x^2}\end{aligned}$$

(g)

$$\begin{aligned}\frac{1+4x^2+6x}{2x-1} &= \frac{(4x^2+1)+6x}{2x-1} \\ &= \frac{(2x+1)(2x-1)+2+6x}{2x-1} \\ &= 2x+1 + \frac{6x+2}{2x-1} \\ &= 2x+1 + \frac{(6x-3)+5}{2x-1} \\ &= 2x+1+3 + \frac{5}{2x-1} \\ &= 2x+4 + \frac{5}{2x-1}\end{aligned}$$

(h)

$$\begin{aligned}\frac{x^2 - 5x + 4}{x^2 + 2x - 3} - \frac{x^2 + 2x}{x^2 + 5x + 6} &= \frac{(x-1)(x-4)}{(x-1)(x+3)} - \frac{x(x+2)}{(x+2)(x+3)} \\ &= \frac{x-4}{x+3} - \frac{x}{x+3} \\ &= \frac{x-4-x}{x+3} \\ &= -\frac{4}{x+3}\end{aligned}$$

3. Show that:

(a) $\sum_{i=1}^N (x_i - \mu_x)^2 = \sum_{i=1}^N x_i^2 - N\mu_x^2$. Hint: Note that $\mu_x = \frac{1}{N} \sum_{i=1}^N x_i$.

$$\begin{aligned}\sum_{i=1}^N (x_i - \mu_x)^2 &= \sum_{i=1}^N (x_i^2 - 2\mu_x x_i + \mu_x^2) \\ &= \sum_{i=1}^N x_i^2 - 2\mu_x \sum_{i=1}^N x_i + \sum_{i=1}^N \mu_x^2 \\ &= \sum_{i=1}^N x_i^2 - 2\mu_x N\mu_x + N\mu_x^2 \\ &= \sum_{i=1}^N x_i^2 - N\mu_x^2.\end{aligned}$$

(b) $\sum_{i=1}^n (a_{i+1} - a_i) = a_{n+1} - a_1$.

$$\begin{aligned}\sum_{i=1}^N (a_{i+1} - a_i) &= \sum_{i=1}^N a_{i+1} - \sum_{i=1}^N a_i \\ &= \sum_{i=1}^N a_i + a_{N+1} - a_1 - \sum_{i=1}^N a_i \\ &= a_{N+1} - a_1.\end{aligned}$$

4. Show that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$\begin{aligned}a^3 - b^3 &= a(a^2 - b^2) + ab^2 - b^3 \\ &= a(a+b)(a-b) + b^2(a-b) \\ &= (a-b)(a^2 + ab + b^2)\end{aligned}$$

5. Differentiate the following functions with respect to x .

(a) $f(x) = 7x^3 - 2x^2 - 5x + 1$
 $f'(x) = 21x^2 - 4x - 5$

$$\begin{aligned}
\text{(b)} \quad & f(x) = 0.7x^{-4} + 1.3 - 3.1x^3 \\
& f'(x) = -2.8x^{-5} - 9.3x^2 \\
\text{(c)} \quad & f(x) = \frac{3x^2 + 1}{2x} \\
& f'(x) = \frac{3}{2} - \frac{1}{2x^2} \\
\text{(d)} \quad & f(x) = \frac{\sqrt{4x+9}}{2} \\
& f'(x) = \frac{1}{\sqrt{4x+9}} \\
\text{(e)} \quad & f(x) = \frac{x^{\frac{1}{3}} - 2}{(x^5 - 2)^3} \\
& f'(x) = \frac{\frac{1}{3}x^{-\frac{2}{3}} \cdot (x^5 - 2)^3 - (x^{\frac{1}{3}} - 2) \cdot 3(x^5 - 2)^2 \cdot 5x^4}{(x^5 - 2)^6} \\
\text{(f)} \quad & f(x) = \ln \left(\frac{x^2}{x^4 + 1} \right) \\
& f'(x) = \frac{x^4 + 1}{x^2} \cdot \frac{2x(x^4 + 1) - x^2 \cdot 4x^3}{(x^4 + 1)^2} = \frac{2}{x} - \frac{4x^3}{x^4 + 1} \\
\text{(g)} \quad & f(x) = e^{x^3+x} \\
& f'(x) = e^{x^3+x} \cdot (3x^2 + 1) \\
\text{(h)} \quad & f(x) = \frac{1}{e^x + e^{-x}} \\
& f'(x) = -1(e^x + e^{-x})^{-2} \cdot (e^x - e^{-x}) = -\frac{e^x - e^{-x}}{(e^x + e^{-x})^2}
\end{aligned}$$

6. Find the all first and second (mixed) partial derivatives of the following functions.

$$\text{(a)} \quad f(x, y) = \ln x \cdot y^2$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{y^2}{x} \\
\frac{\partial^2 f}{\partial x^2} &= -\frac{y^2}{x^2} \\
\frac{\partial f}{\partial y} &= 2y \ln x \\
\frac{\partial^2 f}{\partial y^2} &= 2 \ln x \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{2y}{x}
\end{aligned}$$

$$\text{(b)} \quad f(x, y) = \sqrt{2x - y}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= (2x - y)^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial x^2} &= -(2x - y)^{-\frac{3}{2}} \\
\frac{\partial f}{\partial y} &= -\frac{1}{2}(2x - y)^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial y^2} &= -\frac{1}{4}(2x - y)^{-\frac{3}{2}} \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{1}{2}(2x - y)^{-\frac{3}{2}}
\end{aligned}$$

(c) $f(x, y) = (x + 4y)(e^{-2x} + e^{-3y})$

$$\begin{aligned} f(x, y) &= (x + 4y)(e^{-2x} + e^{-3y}) \\ &= xe^{-2x} + xe^{-3y} + 4ye^{-2x} + 4ye^{-3y}. \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(xe^{-2x}) + \frac{\partial}{\partial x}(xe^{-3y}) + \frac{\partial}{\partial x}(4ye^{-2x}) + \frac{\partial}{\partial x}(4ye^{-3y}) \\ &= (1 \cdot e^{-2x} + x \cdot (-2)e^{-2x}) + e^{-3y} + 4y(-2)e^{-2x} + 0 \\ &= (1 - 2x - 8y)e^{-2x} + e^{-3y}. \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}[(1 - 2x - 8y)e^{-2x}] + \frac{\partial}{\partial x}(e^{-3y}) \\ &= (-2)e^{-2x} + (1 - 2x - 8y) \cdot (-2)e^{-2x} + 0 \\ &= (-2 - 2 + 4x + 16y)e^{-2x} \\ &= 4(-1 + x + 4y)e^{-2x}. \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(xe^{-2x}) + \frac{\partial}{\partial y}(xe^{-3y}) + \frac{\partial}{\partial y}(4ye^{-2x}) + \frac{\partial}{\partial y}(4ye^{-3y}) \\ &= 0 + x(-3)e^{-3y} + 4e^{-2x} + (4e^{-3y} + 4y(-3)e^{-3y}) \\ &= 4e^{-2x} + (4 - 3x - 12y)e^{-3y}. \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}[4e^{-2x} + (4 - 3x - 12y)e^{-3y}] \\ &= 0 + [(-12)e^{-3y} + (4 - 3x - 12y)(-3)e^{-3y}] \\ &= (-12 - 3(4 - 3x - 12y))e^{-3y} \\ &= (9x + 36y - 24)e^{-3y}. \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}[(1 - 2x - 8y)e^{-2x} + e^{-3y}] \\ &= (-8)e^{-2x} + (-3)e^{-3y} = -8e^{-2x} - 3e^{-3y}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}[4e^{-2x} + (4 - 3x - 12y)e^{-3y}] \\ &= -8e^{-2x} + (-3)e^{-3y} = -8e^{-2x} - 3e^{-3y}. \end{aligned}$$

7. For what value of a is the following function continuous for all x ? Is it also differentiable for all x for this value of a ?

$$f(x) = \begin{cases} ax - 1 & \text{if } x \leq 1 \\ 3x^2 + 1 & \text{if } x > 1 \end{cases}$$

Take the limit of the (sub)function $3x^2 + 1$ at $x_0 = 1$

$$\begin{aligned} f(x_0 + h) &= \lim_{h \rightarrow 0} 3(x_0 + h)^2 + 1 \\ f(1 + h) &= 3 \lim_{h \rightarrow 0} (1 + h)^2 + 1 \\ &= 4 \end{aligned}$$

To have continuity, both (sub)functions have to have the same output in the limit. Hence, we equate 4 with the $a - 1$. We receive $a = 5$.

To check for differentiability, we compute the derivative of $g(x) = 5x - 1$, which is $g'(x) = 5$ and $g'(1) = 5$. Since 1 is not in the domain of $h(x) = 3x^2 + 1$, we have to take the limit:

$$\begin{aligned}
 h'(x_0) &= \lim_{h \rightarrow 0} \frac{(3(x_0 + h)^2 + 1) - (3x_0^2 + 1)}{h} \\
 h'(1) &= \lim_{h \rightarrow 0} \frac{(3(1 + h)^2 + 1) - (3 + 1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3(1 + 2h + h^2) + 1 - 3 - 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6h + 3h^2}{h} \\
 &= \lim_{h \rightarrow 0} 6 + 3h \\
 &= 6
 \end{aligned}$$

Since the derivatives at $x_0 = 1$ are not the same, the function can not be differentiable at $x_0 = 1$.